NUMERICAL EVALUATION OF TIME-DOMAIN MOMENTS AND REGULARITY OF MULTIRATE FILTER BANKS

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ABSTRACT

Numerical methods are described for evaluating the time-domain moments and regularity of multirate filter banks. Estimates of the Holder regularity are computed for the continuous functions obtained from the iterated discrete filters. Estimates of the centered moments are also computed for both the discrete filters and continuous functions. These estimates are then used to obtain the vanishing moment numbers. None of the methods require any preprocessing of the filters or *a priori* information about them. Thus, the methods can serve as tests for the evaluation of arbitrary filter banks. Results are presented for various examples.

1. INTRODUCTION

Discrete filters in multirate filter banks are often iterated to approximations of continuous functions. Typically, parameters such as the moments and regularity of the discrete filters and continous functions are studied in an effort to obtain estimates of exact analytic values. However, in computational settings, it would be more meaningful to obtain estimates of iteration-dependent numeric values that better reflect performance in an algorithm with a finite number of iterations of the multirate filter bank.

This report describes new methods with significant advantages for evaluating the numerical behavior of the moments and regularity of iterated filters. Some of these methods were first used in Version 4.0a3 (12-Jan-1994) of the WAVBOX Software Library [3] with results and algorithms described in [4] and [5], respectively. However, they have not yet been published in an archival journal or conference proceedings. Thus, the methods are presented here with examples of results obtained for various multirate filter banks.

2. METHODS

Complete details for all methods described here are elaborated in further detail in [2]. Examples are demonstrated for the DROLD($N;K$) and DROLA($N;K$) filter families [5] which have N coefficients and K roots at $z=-1$. All results reported here were computed with Version 4.5a2 (30-Mar-1998) of the WAVBOX Software Library [3].

2.1. Iterated Filter Banks

Given the $N \times M$ filter bank matrix $\mathbf{F}$ with M band filters $\mathbf{f}_m$, let $\mathbf{Y}^{(0)} = \mathbf{F}$ be the set of initial discrete impulses with matrix $\mathbf{Y} \equiv [y_{nm}]$ and column vectors $\mathbf{y}_m^{(0)} = \mathbf{f}_m$ for $m = 0, 1, \ldots, M-1$. Then let

$$y_m^{(j+1)}[n] = \sum_{k=0}^{N-1} f_0[k] y_m^{(j)}[n - Mk] \tag{1}$$

be the estimates approximating the continuous functions $\psi_m(t_n) \approx y_m^{(J)}[n]$ with discrete samples indexed by n at continuous times $t_n = nM^{-J-1}$ after J iterations.

2.2. Time Domain Moments

Define the q^{th} power weighted discrete and continous time domain centers

$$c_{mq} = \left\langle \left(\sum_{n=0}^{N-1} n|f_m[n]|^q \right) / \left(\sum_{n=0}^{N-1} |f_m[n]|^q \right) \right\rangle \tag{2}$$

$$\gamma_{mq} = \left(\int t|\psi_m(t)|^q \, dt \right) / \left(\int |\psi_m(t)|^q \, dt \right) \tag{3}$$

where the discrete center c_{mq} is integer, the continuous center γ_{mq} is real, and $\langle \cdot \rangle$ in this context denotes rounding to the nearest integer. Identify $J = 0$ and $J > 0$ with the discrete and continous centers respectively for the filters and functions, that is, identify $\psi_m(t_n)$ with $y_m^{(J)}[n]$ for $J > 0$ and $f_m[n]$ with $y_m^{(J)}[n]$ for $J = 0$ as in Section 2.1, and then estimate $c_{mq}^{(0)}$ and $\gamma_{mq}^{(J)}$ numerically. With the weighted centers, define the p^{th} discrete and continous time domain moments

$$k_{mpq} = \sum_{n=0}^{N-1} (n - c_{mq})^p f_m[n] \tag{4}$$

$$\kappa_{mpq} = \int (t - \gamma_{mq})^p \psi_m(t) \, dt \tag{5}$$

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where both k_{mpq} and κ_{mpq} are real. Again, identify $J = 0$ and $J > 0$ with the discrete and continous versions respectively, and then estimate $k_{mpq}^{(0)}$ and $\kappa_{mpq}^{(J)}$ numerically. Numerical integration for the integrals can be performed with a simple quadrature rule such as the trapezoidal rule.

2.3. Vanishing Moments Numbers

For convenience, assume fixed q and J for $\kappa_{mpq}^{(J)}$ and suppress the notation for q and J. Now consider the numerically observed vanishing moments number for $\mathbf{f}_m$ to be the integer ν_m obtained from the sequence of real $\{\kappa_{mp}|p = 0, 1, \ldots\}$ using an absolute zero criterion

$$\nu_m = \min_{p|\chi=1} (p+1)\chi(|\kappa_{mp}| > \epsilon_{\rm abs}) \tag{6}$$

with tolerance $\epsilon_{\rm abs} \approx 0$ such as $\epsilon_{\rm abs} = 1 \times 10^{-4}$, or using a relative jump criterion

$$\nu_m = \min_{p|\chi=1} (p+1)\chi(|\kappa_{m,p+1}/\kappa_{m,p}| > \epsilon_{\rm rel}) \tag{7}$$

with tolerance $\epsilon_{\rm rel} >> 1$ such as $\epsilon_{\rm rel} = 1 \times 10^4$, where $\chi(\cdot)$ is a logical function returning the truth value $\chi \in \{0, 1\}$ for its expression argument. For bandpass or highpass filters, all values of p such that $\chi = 1$ are examined. However, for lowpass filters, all p such that $\chi = 1$ excluding $p = 0$ are examined as necessitated by the fact that the lowpass filter must have a nonvanishing zeroth moment.

2.4. Time Domain Regularity

An iterative estimate of the time domain regularity can be evaluated by applying to Rioul's definition of Holder regularity for subdivision schemes [1] a general procedure for determining the convergence order of a sequence of functions. Assume an arbitrary continuous time function $y(t)$ approximated at iterations j and time points $t_n = nh_j$ by $y^{(j)}[n]$ with error function $e^{(j)}[n] = y(t_n) - y^{(j)}[n]$ where t_n is real, n is integer, and $h_j > 0$ is real $h_j \to 0$ as $j \to \infty$. Let $e^{(j)}[n] = \mathcal{O}(h_j^q)$ as $h_j \to 0$ mean that there exist constants C and h_0 such that $|e^{(j)}[n]| \le Ch_j^q$, $\forall n$, $\forall h_j \le h_0$. Evaluate $e^{(j)}[n]$ for the sequence $h_j = h_0/c^j$ for $j \ge 1$ where $c > 1$ is a constant taken as $c = M$. Define $e_j = \max_n |e^{(j)}[n]|$ so that $e_j \le Ch_j^q$ and $e_{j+1} \le Ch_{j+1}^q$. Derive

$$\frac{e_j}{e_{j+1}} \approx \frac{Ch_j^q}{Ch_{j+1}^q} = \frac{(h_0/c^j)^q}{(h_0/c^{j+1})^q} = c^q \tag{8}$$

for which we can estimate

$$q_j = \frac{\log(e_j/e_{j+1})}{\log(c)} \tag{9}$$

with ideally $q = \lim_{j\to\infty} q_j$.

To account for convergence that is nonmonotonic, use smoothers such as the median as well as the lower and upper bounds to define the estimates

$$\underline{q}_j = \min\{q_i \mid i = j_0, j_0+1, \ldots, j\} \tag{10}$$
$$\hat{q}_j = \mathrm{med}\{q_i \mid i = j_0, j_0+1, \ldots, j\} \tag{11}$$
$$\overline{q}_j = \max\{q_i \mid i = j_0, j_0+1, \ldots, j\} \tag{12}$$

that provide checks on the behavior of the convergence.[1] After J iterations, obtain the final estimate $\hat{q}_J$ bounded below by $\underline{q}_J$ and above by $\overline{q}_J$. The number of iterations J can be determined by a convergence criterion such as

$$|\hat{q}_j - \hat{q}_{j-1}| < \epsilon \tag{13}$$

for some absolute error tolerance ϵ or else J can be fixed by a predetermined value. This approach provides an iterative method to estimate the convergence order q without assuming its value *a priori* and without knowing the constant C.

Now let $\mathbf{\Delta}^p$ be the finite difference operator of order p. Assume for $y(t)$ the regularity $\rho = p + q$ with integer p and real q. Then use the method described above to estimate q in the sequence

$$e_j = \max_n |\mathbf{\Delta}^p y^{(j)}[n]|/h_j^p \le h_j^q \tag{14}$$

by testing iterates $y^{(j)}[n]$ with an appropriate known p. An effective algorithm can be implemented as a search for p_k over $k = 1, 2, \ldots$ where for each p_k a cycle of iterations over $j = 0, 1, \ldots, J_k$ is performed. Equation 13 provides a test of convergence of $\hat{q}_j$ which determines J_k for a given cycle with p_k at iteration k. Now let

$$\rho_k = p_k + \hat{q}_{J_k} \tag{15}$$

denote the $k^{\rm th}$ regularity estimate. Values for p_{k+1} can be set from those for p_k by the recursion

$$p_{k+1} = \begin{cases} p_k + 1 & \text{if } \lceil \rho_k \rceil + 1 > p_k \\ p_k - 1 & \text{if } \lceil \rho_k \rceil + 1 < p_k \end{cases} \tag{16}$$

with initialization $p_1 = 2$ and termination if

$$\lceil \rho_k \rceil + 1 = p_k \tag{17}$$

or if k exceeds a predetermined number of iterations.

For an experiment comparing these estimates with those of Rioul, let $\mathrm{rlb}(\cdot)$ and $\mathrm{rub}(\cdot)$ denote respectively his iterative estimate for the lower bound [1, eqn.11.1] and noniterative estimate for the upper bound [1, eqn.13.1]. Both of his estimates require that the filter roots at $z = -1$ be deconvolved prior to estimation of the filter's regularity. Thus, for the comparison experiment, let $\mathbf{f}$ and $\mathbf{g}$ be corresponding exact filters (both vectors of coefficients constructed exactly from known roots) with and without the roots at $z = -1$, respectively. Let $\hat{\mathbf{g}}$ be the filter $\mathbf{g}$ estimated from $\mathbf{f}$ by testing for and deconvolving an unknown number of roots at $z = -1$.

[1] The bounds are computed for $j \ge j_0$ to allow for initialization transients, for example with $j_0 = 2$.

3. RESULTS

Table 1 lists regularity estimates obtained with $J \leq 5$ for the DROLD($N;K$) family of filters. All of the estimates $\underline{\rho}(\mathbf{f}) \leq \hat{\rho}(\mathbf{f}) \leq \overline{\rho}(\mathbf{f})$ remain stable for $2 \leq K \leq 24$. In contrast, both of the estimates rlb($\hat{\mathbf{g}}$) $\leq$ rub($\hat{\mathbf{g}}$) require preprocessing to obtain $\hat{\mathbf{g}}$ from $\mathbf{f}$ and they become unstable for $21 \leq K \leq 24$. Moreover, the iterative $\underline{\rho}(\mathbf{f})$ converges faster to tighter lower bounds and outperforms the iterative rlb($\hat{\mathbf{g}}$) for all $2 \leq K \leq 24$. Although the noniterative rub($\mathbf{g}$) remains stable for $2 \leq K \leq 24$, it does require *a priori* knowledge of $\mathbf{g}$ and thus cannot be applied to an arbitrary unknown filter $\mathbf{f}$. See [2] for examples demonstrating $\hat{\rho}$ for bandpass and highpass filters.

Figures 1, 2, and 3 display respectively the $q = 2$ centered time-domain moments and vanishing moment numbers at $J = 0$ and $J = 2$ for the DROLA(32;16) filter bank. In Figure 1 with $|\kappa_{mp}|$ as a function of p (scalets $\mathbf{f}_0$ on left, wavelets $\mathbf{f}_1$ on right), the discrete moments (top curve in each subfigure) for $\mathbf{f}_m$ are clearly distinct from the iterative estimates of the continuous moments for $\mathbf{f}_m$ with $j = 1, 2, 3$. In Figures 3 and 4 for the vanishing moment numbers for the scalets and wavelets at $J = 2$, using the absolute zero criterion (left subfigure in each), $\boldsymbol{\nu} = [0, 12]$ was observed for DROLD(32;16) while $\boldsymbol{\nu} = [0, 16]$ was observed for DROLA(32;16), but $[0, 16]$ was expected for both. However, using the relative jump criterion (right subfigure in each), $\boldsymbol{\nu} = [0, 16]$ was observed for both as expected.

4. DISCUSSION

Numerical methods have been presented for estimating time-domain centers, moments, vanishing moment numbers, and regularity of all filters in a filter bank. The methods do not need to be restricted to the lowpass filter only. Nor do they require preprocessing of the filter or *a priori* information about the filter. Thus, they are applicable to the evaluation of arbitrary multirate filter banks. Examples of results in addition to those reported here can be found in [2] for a variety of M-band filter banks with $M \geq 2$. For the determination of the vanishing moment numbers, the relative jump criterion was found to reveal the expected result in situations where the absolute zero criterion did not. However, it is the estimate obtained with the absolute zero criterion that reflects the actual number of *effective* vanishing moments impacting the numerical computing application in practice. For the determination of the regularity, the iterative method presented here does not insure monotonic convergence. However, it does provide faster convergence than the iterative method described by Rioul [1]. Moreover, it has the significant advantage that the roots at $z = -1$ do *not* need to be deconvolved prior to evaluation of the regularity estimate.

5. REFERENCES

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Table 1: Time-domain regularity for DROLD$(N;K)$

K	rlb($\hat{\mathbf{g}}$)	rub($\mathbf{g}$)	rub($\hat{\mathbf{g}}$)	$\underline{\rho}(\mathbf{f})$	$\hat{\rho}(\mathbf{f})$	$\overline{\rho}(\mathbf{f})$
2	0.400	0.550	0.550	0.550	0.550	0.550
3	0.887	1.088	1.088	1.078	1.089	1.105
4	1.312	1.618	1.618	1.481	1.608	1.663
5	1.457	1.969	1.969	1.834	1.954	2.102
6	1.527	2.189	2.189	2.132	2.184	2.201
7	1.658	2.460	2.460	2.326	2.485	2.814
8	1.819	2.761	2.761	2.732	2.836	2.968
9	1.933	3.074	3.074	2.816	3.131	3.422
10	2.039	3.381	3.381	3.254	3.363	3.474
11	2.128	3.603	3.603	3.086	3.610	3.901
12	2.194	3.833	3.833	3.656	3.773	4.095
13	2.267	4.073	4.073	3.574	3.952	4.503
14	2.342	4.317	4.317	4.083	4.309	4.723
15	2.413	4.558	4.558	4.417	4.466	4.486
16	2.473	4.791	4.791	4.575	4.801	4.905
17	2.536	5.014	5.014	4.629	4.978	5.069
18	2.599	5.239	5.240	4.911	5.197	5.649
19	2.628	5.465	5.441	5.357	5.450	5.981
20	2.796	5.691	5.759	5.401	5.573	6.286
21	-5.093	5.916	-0.638	5.833	5.960	6.199
22	5.982	6.138	8.767	5.952	6.167	6.269
23	3.161	6.360	6.592	6.269	6.495	6.766
24	10.185	6.581	12.622	6.362	6.638	7.208

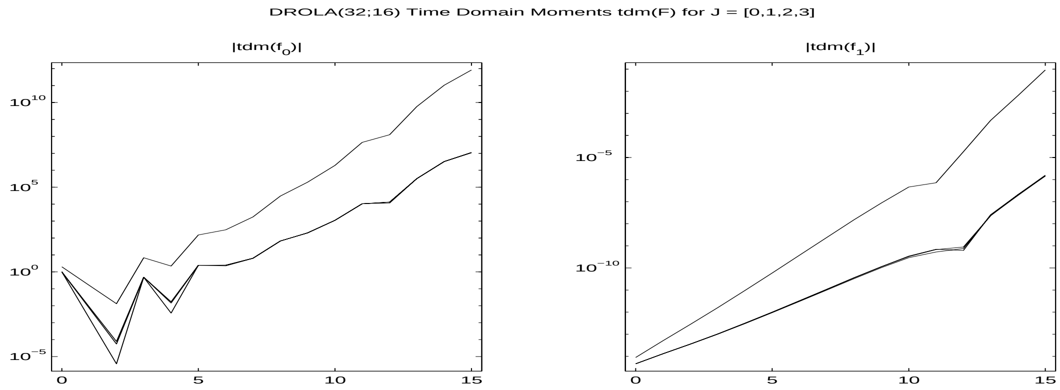

Figure 1: Time domain moments for DROLA(32;16).

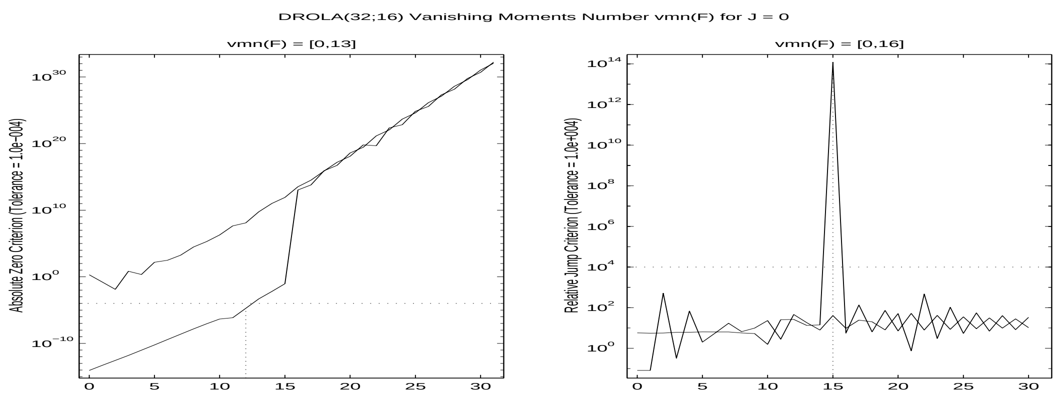

Figure 2: Vanishing moment numbers at $J = 0$ for DROLA(32;16).

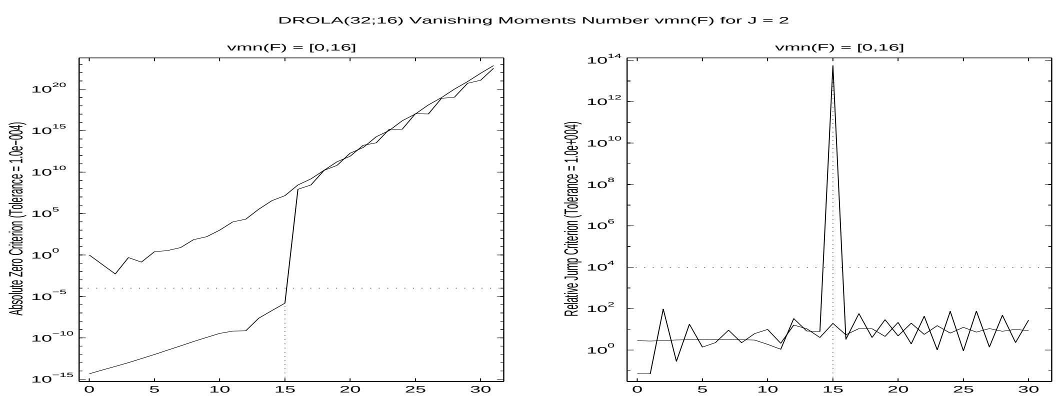

Figure 3: Vanishing moment numbers at $J = 2$ for DROLA(32;16).

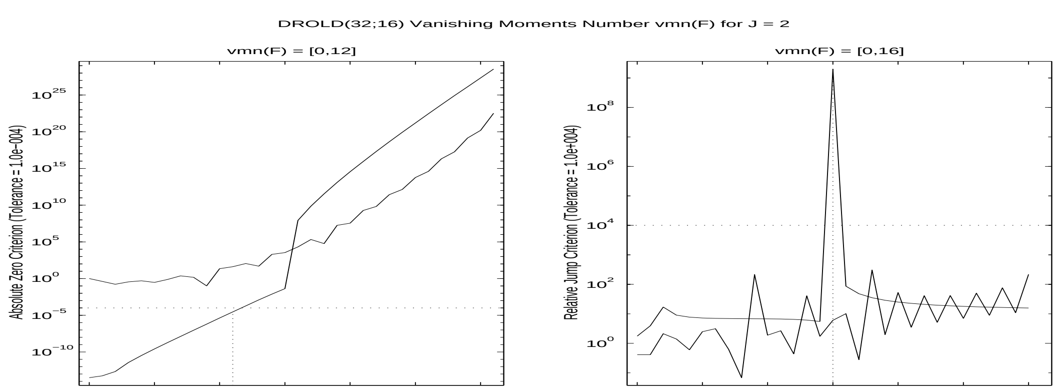

Figure 4: Vanishing moment numbers at $J = 2$ for DROLD(32;16).